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Anomalous phase synchronization in populations of nonidentical oscillators

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We report the phenomenon of anomalous phase synchronization in interacting oscillator systems with randomly distributed parameters. We show that coupling is first able to enlarge the frequency disorder leading to maximal decoherence for intermediate levels of coupling strength before reaching synchronization. Anomalous synchronization arises when the natural frequency covaries with nonisochronicity and allows for synchronization control by adjustment of system parameters.

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The study of interacting oscillator systems is one of the fundamental problems in nonlinear dynamics [1,2]. In practice, it is inevitable that the oscillators are nonidentical and vary in their natural frequencies. Of special interest is the phenomenon of phase synchronization in which coupling can overcome the dispersal of natural frequencies and the oscillators are mutually entrained to a common locking frequency [3]. Phase synchronization is an ubiquitous phenomenon and arises in many areas of physics and living systems. It has been observed in coupled pairs of oscillators, in one- or two-dimensional lattices and in ensembles of globally coupled limit cycle [4] and chaotic oscillators [5,6]. Biological examples include synchronous flashing fireflies [7], neural networks [8], the cardiorespiratory system [9], and oscillating population numbers [6,10]. Usually, the interaction leads to locking of the oscillator frequencies, but coupling may have different effects including oscillation death [11,12], desynchronization via short-wavelength bifurcation [13], or dephasing with bursts of amplitude change [14].

In this paper we investigate the effect of weak interaction on the frequency distribution in a set of nonidentical oscillators. We show that the usual transition to phase synchronization can be strongly modified when the disorder is affecting two characteristics of the system simultaneously. Under these assumptions, we show a mechanism for coupling induced desynchronization where the interaction does not immediately lead to an increase of synchrony in the network but is first able to enlarge the natural frequency disorder. This phenomenon appears universally when nonisochronicity and natural frequency of the oscillators have a positive covariance. The effect is of potential use for engineering applications because it allows for synchronization control: with an appropriate choice of oscillator parameters it is possible to either enhance or inhibit the synchronization in the ensemble. Similar strategies can easily be used in living systems and therefore the effect is of considerable importance in a variety of physical and biological applications.

We consider a system of N coupled oscillators

$$\dot{\mathbf{x}}_i=\mathbf{F}(\mathbf{x}_i,\chi_i)+\frac{\epsilon}{m}C\sum_{j\in N_i}(\mathbf{x}_j-\mathbf{x}_i), \quad i=1 \quad N. \tag{1}$$

In the absence of coupling each autonomous oscillator $\mathbf{x}_i \in \mathrm{R}^n$ follows its own local dynamics $\dot{\mathbf{x}}_i=\mathbf{F}(\mathbf{x}_i,\chi_i)$, which we assume to be either a limit cycle or phase coherent chaos. Accordingly, each oscillator is characterized by a well-defined natural frequency $\omega_i=\overline{\dot{\theta}_i(t)}$ given as the long time average of phase velocity [3].

Quenched disorder is imposed onto the system by assigning to each oscillator i an independent set of control parameters $\chi_i=(a_i,b_i\ldots)$ which affect the natural frequency $\omega_i=\omega(\chi_i)$. This natural disorder in control parameters leads to a frequency mismatch between the oscillators, which we also refer to as frequency disorder. The oscillators are then coupled with strength ϵ over a predefined set N_i of m neighbors and using the diagonal coupling matrix $C=\mathrm{diag}(c_1,c_2,\ldots,c_n)$.

Synchronization then arises as an interplay of the interaction and the frequency mismatch between the oscillators. Thereby, in general, all frequencies $\Omega_i=\Omega_i(\epsilon)$ will be detuned from the natural frequency, i.e., $\omega_i=\Omega_i(0)$. It is convenient to measure the amount of synchronization with the standard deviation of all oscillator frequencies, $\sigma(\epsilon)$. Phase synchronization refers to the fact that with sufficient coupling strength $\epsilon>\epsilon_c$ all oscillators rotate with the same frequency and implies $\sigma(\epsilon)=0$.

We start by comparing the transition to synchronization in ensembles of two phase coherent chaotic oscillators, namely, the Rössler system [5]

$$\dot{x}=-b_iy_i-z_i, \quad \dot{y}=b_ix_i+a_iy_i, \quad \dot{z}=0.4+(x_i-8.5)z_i \tag{2}$$

and the following chaotic predator-prey model that has been introduced in Ref. [6] to describe large scale synchronization effects in ecological systems:

$$\dot{x}=x_i-1.5-0.1x_iy_i, \quad \dot{y}=-b_iy_i+0.1x_iy_i-0.6y_iz_i$$

$$\dot{z}=-10z_i+0.1+0.6y_iz_i. \tag{3}$$

Both systems have a free parameter b_i which is taken for each oscillator from the same statistical distribution. Despite the fact that both models have a very similar attractor topology [6], we find fundamental differences in their response to the interaction (see Fig. 1). In the ensemble of Rössler systems we observe the usual onset of synchronization where the frequency disorder $\sigma(\epsilon)$ is a monotonically decreasing function of coupling strength. In contrast, the ensemble of foodweb models (3) shows a totally different behavior. Here,

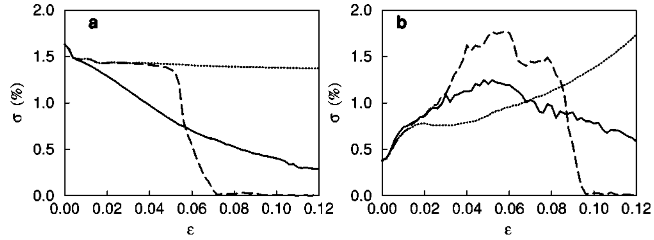

FIG. 1. Standard deviation of frequencies $\sigma(\epsilon)$ in a population of 500 coupled oscillators; (a) Rössler system (2) with $a_i=0.15$ and (b) foodweb model (3). Oscillators have been coupled in the y variable, $C=\mathrm{diag}(0,1,0)$, with strength ϵ to either next neighbors in a ring with periodic boundaries (solid line), with global coupling (dashed line), or using approximation (14) (dotted line). Parameters b_i were taken as uniformly distributed random numbers in the range 0.97 ± 0.025. The phase evolution of each oscillator was determined by counting the maxima of $y_i(t)$ [5].

with increasing coupling strength $\sigma(\epsilon)$ is first amplified and synchronization sets in only for much larger coupling with a maximal decoherence for intermediate values of ϵ. We denote this counterintuitive increase of disorder with coupling strength as *anomalous phase synchronization* (APS). Note, that here dephasing sets in without threshold, which differs totally from other coupling induced effects where instabilities arise only when coupling exceeds a critical level [11].

We have tested the robustness of APS in a large number of numerical simulations. We have found APS in the ecological model (3) for various coupling topologies (one- and two-dimensional lattices or global coupling) and for different ensemble sizes. In general, coupling induced decoherence is more distinct with a larger number of oscillators, N, or next neighbors m. It can already be observed in two coupled oscillators. APS appears when disorder is realized by various statistical distributions, largely independent of the width of the dispersal, and it retains also in chains with linearly increasing parameters b_i. Furthermore, the effect is robust to the presence of noise and is independent of the initial conditions.

In the limit of weak interactions the dynamics of system (1) can be written in the form of phase equations

$$\dot{\theta}_i=\omega_i+\frac{\epsilon}{m}\sum_{j\in N_i}\Gamma_{ij}(\theta_j-\theta_i), \qquad (4)$$

where the interaction function Γ_{ij} is given by [2,12,15]

$$\Gamma_{ij}(\Delta\theta_{ij})=\frac{1}{2\pi}\int_{\phi=0}^{2\pi}Z_i(\vartheta)p_{ij}(\vartheta,\Delta\theta_{ij})d\vartheta. \qquad (5)$$

Here, $p_{ij}(\vartheta,\delta\theta)$ describes the perturbation of the state of oscillator i with phase ϑ due to the interaction with oscillator j at phase $\vartheta+\Delta\theta$, and the sensitivity vector $Z_i(\vartheta)$ gives the phase shift of oscillator i after the perturbation.

We first explain APS for an ensemble of globally coupled *Landau-Stuart oscillators* as a widely studied canonical model for weakly nonlinear limit-cycle systems [2,4,12]

$$\dot{z}_i=(1+i\eta_i)z_i-(1+i\alpha_i)|z_i|^2z_i+\frac{\epsilon}{N}\sum_j (z_j-z_i). \qquad (6)$$

Rewriting the complex variable z_i in polar coordinates (r_i,θ_i) and considering weak disorder, Eq. (6) becomes

$$\dot{r}_i=r_i(1-r_i^2)+\frac{\epsilon}{N}r_i\sum_j [\cos(\theta_j-\theta_i)-1],$$

$$\dot{\theta}_i=\eta_i-\alpha_i r_i^2+\frac{\epsilon}{N}\sum_j \sin(\theta_j-\theta_i). \qquad (7)$$

Here, the natural frequency ω_i is determined by the difference of the amplitude independent rotation speed η_i and an amplitude dependent term $\alpha_i r_i^2$ that reflects the nonisochronicity or shear of phase flow around the limit cycle [2]. After relaxation of amplitudes the system can be written in the generic form (4) with $\Gamma_{ij}(\theta_j-\theta_i)=\sin(\theta_j-\theta_i)+\alpha_i[1-\cos(\theta_j-\theta_i)]$ and $\omega_i=\eta_i-\alpha_i$.

Assuming for small ϵ that the oscillators rotate independently we can approximate $\Sigma_j\cos(\theta_j-\theta_i)\approx\Sigma_j\sin(\theta_j-\theta_i)\approx 0$. Thus, we arrive at $\dot{r}_i=r_i(1-r_i^2)-\epsilon r_i$ and the amplitude, on average, is perturbed to $r_i^2=1-\epsilon$. As a result, the mean frequency becomes a function of coupling strength

$$\Omega_i(\epsilon)=\eta_i-\alpha_i(1-\epsilon)=\omega_i+\epsilon\alpha_i. \qquad (8)$$

It is important to note that Eq. (6) has to be understood as an effective equation that describes the amplitude and phase dynamics of the specific system (1) with two characteristic constants ω_i and α_i, which, in general, will both be functions of the system parameters $\omega_i=\omega(\chi_i)$ and $\alpha_i=\alpha(\chi_i)$. As a consequence, α_i and ω_i are implicitly related $\alpha_i=\alpha_i(\omega_i)$ and thus can not be treated as independent parameters. After simple calculation we obtain the standard deviation of the ensemble frequencies in Eq. (8),

$$\sigma(\epsilon)=\sigma_\omega+\frac{\epsilon}{\sigma_\omega}\mathrm{Cov}(\omega_i,\alpha_i)+O(\epsilon^2), \qquad (9)$$

where σ_ω is the standard deviation of ω_i. Here, the frequency disorder up to first order in ϵ increases with the covariance of natural frequency and nonisochronicity of all oscillators in the ensemble. Thus, we expect an anomalous enlargement when α_i increases with ω_i.

For a small range of frequencies we can linearize the relation $\alpha_i(\omega_i)$ in first order

$$\alpha_i=\alpha(\omega_i)=k\omega_i+\tilde{\alpha}. \qquad (10)$$

The coefficient k measures the relation between nonisochronicity and natural frequency. Now, Eq. (9) becomes

$$\sigma(\epsilon)=(1+k\epsilon)\sigma_\omega+O(\epsilon^2). \qquad (11)$$

Figure 2 illustrates these results with a numerical simulation of ten coupled oscillators (6) where the disorder has been distributed according to Eq. (10). If $k=0$, the system shows the usual transition to synchronization, i.e., without

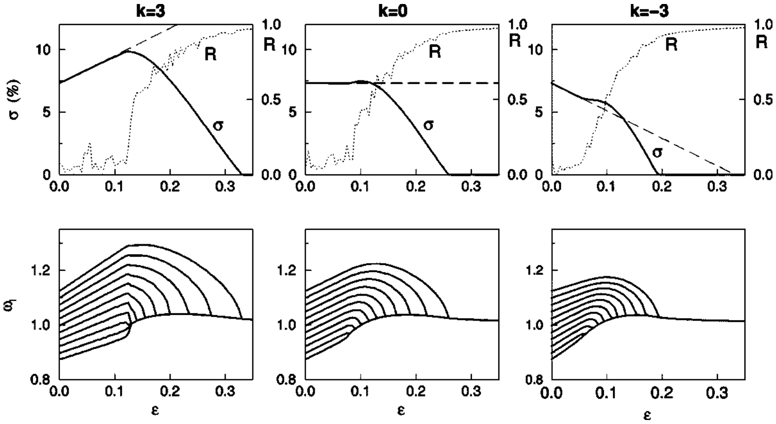

FIG. 2. Transition to synchronization in an ensemble of ten globally coupled Landau-Stuart oscillators (6) with $\alpha_i = k(\omega_i - \langle\omega_i\rangle) + \langle\omega_i\rangle$ and $\omega_i = 1 \pm 0.125$. (Left) $k=3$, anomalous synchronization; (center) $k=0$, usual synchronization; (right) $k=-3$, enhanced synchronization. (Top) Standard deviation of the ensemble frequencies $\sigma(\epsilon)$; numerical simulation (solid line) and analytical result (11) (dashed line). Further indicated is the amplitude of the complex order parameter R (dotted line). (Bottom) frequencies $\Omega_i(\epsilon)$ of individual oscillators.

anomalous behavior. However, when nonisochronicity and natural frequency are related, $k \neq 0$, the system exhibits APS similar to the chaotic foodweb model (3). The mechanism is obvious from Fig. 2, where for small ϵ the frequencies $\Omega_i(\epsilon)$ change linearly with ϵ (8), with a slope $\alpha_i(\omega_i)$ that depends on the natural frequency. Thus, if $k>0$ and α_i increases with ω_i we first observe an effective enlargement of the frequency differences [see Fig. 2(a)]. In contrast, if $k<0$, the opposite effect happens and synchronization is immediately strongly enhanced.

Despite the fact that Eq. (11) has been derived for very small ϵ, APS prevails in the large coupling regime. As a consequence, in Fig. 2 the synchronization threshold ϵ_c changes significantly with k, a fact that is also reflected by the amplitude of the complex order parameter $R = |\langle e^{i\theta_j}\rangle|$ [2].

In the case of only two phase oscillators (6), the full transition to synchronization can be described analytically. After relaxation of amplitudes in Eq. (7) the phase difference $\phi = \theta_2 - \theta_1$ is determined by [12]

$$\dot{\phi} = \Delta\omega - \epsilon[2\sin\phi + \Delta\alpha(\cos\phi - 1)]. \tag{12}$$

Then the mean frequency difference can be calculated as $\Delta\Omega = \langle\dot{\phi}\rangle = (1/2\pi\int_0^{2\pi} d\phi/\dot{\phi})^{-1}$, which results in

$$\Delta\Omega(\epsilon) = \sqrt{\Delta\omega^2 + 2\epsilon\Delta\omega\Delta\alpha - 4\epsilon^2}. \tag{13}$$

For $\Delta\alpha = 0$ this expression reduces to the well known beat frequency of two coupled phase oscillators. However, when both oscillators differ in nonisochronicity, Eq. (13) exhibits APS in very good agreement with the numerical simulations (Fig. 3).

After these explanations for universal limit cycle models we return to the general, possibly chaotic, system (1). Using similar ideas, we can describe the frequency disorder in the regime of weak global coupling. First note that in the absence of coupling the oscillators are rotating independently of each other. With the onset of weak coupling we can assume that the oscillators remain to be independent. Thus, for $\epsilon \ll 1$ and in the thermodynamic limit the ensemble average is constant in time, $\langle\mathbf{x}_j\rangle = \boldsymbol{\xi}$, and we can approximate the interacting system as a system of N uncoupled oscillators with modified dynamics

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i, \chi_i) - \epsilon C(\mathbf{x}_i - \boldsymbol{\xi}). \tag{14}$$

Consequently, for $\epsilon \ll 1$ the frequency detuning of each oscillator i depends only on its own parameters χ_i,

$$\Omega_i = \Omega(\chi_i, \epsilon) \approx \omega(\chi_i) + \epsilon\kappa(\chi_i) + O(\epsilon^2). \tag{15}$$

Here, $\kappa(\chi_i)$ describes the frequency response of each oscillator to the onset of interaction and by comparison with Eq. (8) it can be identified with the nonisochronicity α_i in weakly nonlinear systems. Another way to derive Eq. (15) stems from Eq. (4) where again using a random phase approximation and after averaging we find for $\epsilon \ll 1$,

$$\kappa(\chi_i) = \frac{1}{m}\sum_{j\in N_i}\frac{1}{2\pi}\int_0^{2\pi}\Gamma_{ij}(\Delta\theta_{ji})d\Delta\theta_{ji}. \tag{16}$$

Thus, in principle, the characteristics ω_i and κ_i can be calculated from the basic equations (1) and these are given as functions of the control parameters of the system

$$F:\Sigma \to \mathrm{R}^2, \quad \chi_i \mapsto \omega(\chi_i), \kappa(\chi_i) . \tag{17}$$

Here, $\Sigma \subseteq \mathrm{R}^l$ denotes the parameter space for each individual oscillator and $\chi_i = (a_i, b_i, \ldots)$. The crucial fact is that, in general, $\omega(\chi_i)$ and $\kappa(\chi_i)$ are not functionally independent. In analogy to Eq. (9) the frequency disorder is determined by

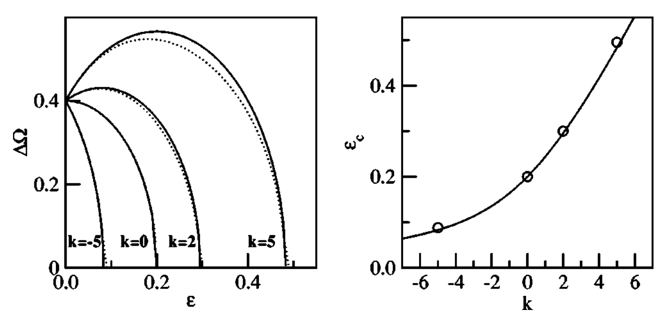

FIG. 3. Left: Frequency difference of two coupled phase oscillators (6) as a function of coupling strength for different values of k when $\Delta\alpha = k\Delta\omega$ ($\omega_1 = 1.2, \omega_1 = 0.8$). Solid line: analytical result (13). Dotted line: numerical simulation. Right: synchronization threshold ϵ_c as a function of k. Solid line: analytical result. Circles: numerical result.

the covariance between ω_i and κ_i in the ensemble. If in the whole disordered ensemble the parameters are distributed in a subset $S \subset \Sigma$, then anomalous enlargement appears if $\mathrm{Cov}[F(S)] > 0$.

Following these ideas, we now show that anomalous behavior can arise in the ensemble of Rössler systems (2). Recall that by varying only the b_i anomalies are absent (Fig. 1). Now we allow also for variations in the second parameters a_i. The problem then is to distribute the disorder in such a way among system parameters that ω_i and κ_i are both increasing or decreasing functions of χ_i. In the simplest scenario we demand that a_i and b_i are linearly related by $a_i - \langle a \rangle = k(b_i - \langle b \rangle)$. Thus varying k, we effectively try different directions in parameter space. And indeed as is shown in (Fig. 4) we observe anomalous synchronization for $k > 0$, usual synchronization for $k = 0$, and enhanced synchronization for $k < 0$.

In conclusion, we have described the effect of anomalous phase synchronization in ensembles of nonidentical oscillators where the usual transition to synchronization is strongly modified, either enhancing or inhibiting synchrony. APS appears because the interaction perturbs the oscillators away from their attractors. This brings the nonisochronicity of oscillation into play. Disorder enlargement occurs if nonisochronicity covaries with the natural frequency. APS is also relevant in the large coupling regime and strongly controls the synchronization threshold. Thus, the effect is important for applications. Similar to Fig. 4, an experimentalist can modify the synchronization properties of a given system if he has the freedom to adjust individual oscillator parameters.

Beyond its importance for the theory of synchronization, APS has wider implications for biological systems that are typically characterized by large amounts of inherent disorder. In many cases strong synchronization is desirable for biological reasons [3]. Therefore, it is quite possible that evolution has made use of APS by selecting individuals with correlated system parameters to speed up synchronization and to compensate for the natural heterogeneity of all living environments. On the other hand, there are situations where synchronization is regarded as dangerous. For example, it is known that synchronization of fluctuating population numbers is strongly connected to the risk of global species extinction [10]. In this respect the anomalous synchrony inhibition in ecological models has important consequences for conservation ecology.

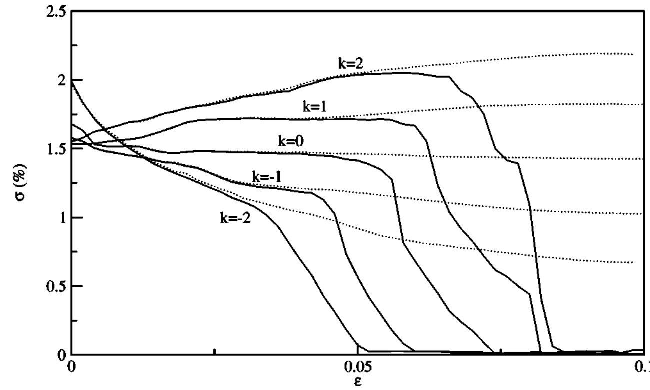

FIG. 4. Possibility of anomalous behavior in the Rössler system which is achieved by simultaneous variation of two parameters. Plotted is the frequency disorder $\sigma(\epsilon)$ as a function of coupling strength ϵ in an ensemble of 500 globally coupled Rössler oscillators (solid lines) and using approximation (14) (dotted lines). Parameter values b_i are taken as in Fig. 1 and $a_i = k(b_i - \langle b \rangle) + \langle a \rangle$ with $\langle b \rangle = 0.97$ and $\langle a \rangle = 0.15$.

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